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AN IMPROVEMENT METHOD FOR SOLVING THE INDEX PROBLEM FOR STURM-LIOUVILLE EIGENVALUES

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Abstract

An improvement is given to the known solution of the index problem for Sturm-Liouville eigenvalues for coupled boundary conditions. The algorithm corresponding the solution is discussed, and numerous numerical examples illustrate the theoretical results and show that the algorithm is valid.

1. Introduction

Consider a self-adjoint Sturm-Liouville problem (SLP), i.e., the spectral problem consisting of a Sturm-Liouville equation (SLE)

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$$-(fy')' + qy = \lambda wy \text{ on } (a, b), \tag{1.1}$$

and a self-adjoint boundary condition (BC), where

$$-\infty < a < b < +\infty, \quad 1/f, q, w \in L_{loc}((a, b), \mathbb{R}), \quad w > 0 \text{ a.e. on } (a, b), \tag{1.2}$$

and $\lambda \in \mathbb{C}$ is the so-called *spectral parameter*. Here, for an interval $J \in \mathbb{R}$, we denotes by $L_{loc}(J, \mathbb{R})$ the set of real-valued functions on J, which are Lebesgue integrable on all compact subintervals of J.

We focus our attention on the index problem for Sturm-Liouville eigenvalues, especially for eigenvalues of SLP with coupled boundary conditions. The search of such index problem for coupled boundary conditions has not a history longer, see [9, 10] for little bit more information.

In addition to being interesting by their own, such index problem has several applications. First, they yield some global information about the eigenvalue, such as estimate of the range of n-th eigenvalue. Second, such index problems are essential ingredients needed after solving the eigenvalues in a chosen interval. Third, they imply an algorithm for numbering the eigenvalues for coupled self-adjoint BCs, when the spectrum is unbounded from both below and above.

In the paper [10], based on relations among β_0, $\beta_{0,K}$, β_1, $\beta_{1,K}$, a simple solution of the index problem has been obtained, and it works in the same way for both the spectrum can be either bounded in one direction, or unbounded from both below and above, this simple solution of the index problem has been directly implemented on a computer, see [9]. In this paper, using α_0, $\alpha_{0,K}$, α_1, $\alpha_{1,K}$ as defined in Definition 1, the index problem is solved likewise, moreover, the method can predigest the process solving the index problem. In this paper, a simple method of the index problem is discussed, see Theorem 1. An idea of the algorithm based on Theorem 1 is presented, and numerous examples are given. We would like to emphasize that 9 possibilities in [9] are sorted into 3 kinds, furthermore, the method solving is terse than one in [9].

This paper is organized as follows. In Section 2, we introduce our notation and present the above mentioned simple solution of the index problem. Section 3 is devoted to the idea corresponding algorithm, while examples are given in Section 4.

Throughout this paper, we fix the SLE (1.1) satisfying (1.2), and all BCs considered are self-adjoint.

2. Notation and Basic Results

Let $\mathrm{SL}(2, \mathbb{R})$ be the set of real matrices in dimension 2 and having determinant 1. When a capital Latin or Greek letter stands for a matrix, the entries of the matrix are always denoted by the corresponding lower case letter with two subindices. For example, the entries of a matrix K are k_{ij}'s.

By the integrability conditions in (1.2), all solutions y of (1.1) and their quasi-derivatives fy' have finite limits at both a and b. For every $\lambda \in \mathbb{C}$, let $\phi_{11}(\cdot, \lambda)$ and $\phi_{12}(\cdot, \lambda)$ be the solutions of (1.1) is determined by the initial conditions

$$\phi_{11}(a, \lambda) = 1, \ (f\phi_{11}')(a, \lambda) = 0, \quad \phi_{12}(a, \lambda) = 0, \ (f\phi_{12}')(a, \lambda) = 1. \tag{2.1}$$

Then, they form a fundamental set of solutions for (1.1). We denote $f\phi_{11}'$ and $f\phi_{12}'$ by ϕ_{21} and ϕ_{22}, respectively. Set

$$\Phi(t, \lambda) = \begin{pmatrix} \phi_{11}(t, \lambda) & \phi_{12}(t, \lambda) \\ \phi_{21}(t, \lambda) & \phi_{22}(t, \lambda) \end{pmatrix}, \quad t \in [a, b], \ \lambda \in \mathbb{C}. \tag{2.2}$$

Then, $\Phi(t, \lambda)$ satisfies the matrix form of (1.1), i.e.,

$$\Phi'(t, \lambda) = \begin{pmatrix} 0 & 1/f(t) \\ q(t) - \lambda w(t) & 0 \end{pmatrix} \Phi(t, \lambda) \text{ on } (a, b), \tag{2.3}$$

and the initial condition $\Phi(a, \lambda) = I$. We call Φ the *fundamental solution matrix* of (1.1). Note that $\Phi(t, \lambda) \in \mathrm{SL}(2, \mathbb{R})$ for $t \in [a, b]$ and $\lambda \in \mathbb{R}$.

For any solution y of (1.1), let

$$Y(t) = \begin{pmatrix} y(t) \\ (fy')(t) \end{pmatrix}, \quad t \in [a, b]. \tag{2.4}$$

Then, BCs are specified by algebraic systems of the form

$$AY(a) + BY(b) = 0, \tag{2.5}$$

where A and B are 2 by 2 complex matrices such that the 2 by 4 matrix $(A \mid B)$ has rank 2. Note that equivalent algebraic systems define the same BC. Bold faced capital Latin letters, such as $\boldsymbol{A}$, are used to stand for BCs.

Each separated self-adjoint BC has its standard form

$$\cos\alpha \cdot y(a) - \sin\alpha \cdot (fy')(a) = 0 = \cos\beta \cdot y(b) - \sin\beta \cdot (fy')(b) \tag{2.6}$$

with $\alpha \in [0, \pi)$ and $\beta \in (0, \pi]$. We denote this BC by $\boldsymbol{S}_{\alpha,\beta}$. Moreover, $\boldsymbol{S}_{\alpha,\beta}$ can be defined for any $\alpha, \beta \in \mathbb{R}$, and ranges of α and β different from $[0, \pi)$ and $(0, \pi]$ will also be used later.

Every coupled self-adjoint BC can be written into the form

$$Y(b) = e^{i\gamma} KY(a) \tag{2.7}$$

with $\gamma \in [0, \pi)$ and $K \in \mathrm{SL}(2, \mathbb{R})$. Sometimes, it is convenient to allow other ranges of γ, such as $(-\pi, \pi)$ and $\mathbb{R}$.

For any non-trivial real solution of (1.1), there are two unique absolutely continuous functions ρ and θ on $[a, b]$ such that $\rho(t, \lambda) \neq 0$, for all $t \in [a, b]$, and

$$y = \rho \sin\theta, \quad fy' = \rho\cos\theta, \quad 0 \leq \theta(a, \lambda) < \pi. \tag{2.8}$$

The function θ is called the *Prüfer angle* of the solution y. The zeros of y in $[a, b]$ are exactly the points of $[a, b]$, where θ attains an integer multiple of π. Note that y satisfies the self-adjoint BC $\boldsymbol{S}_{\alpha,\beta}$, if and only if

$$\theta(a, \lambda) = \alpha, \quad \theta(b, \lambda) = \beta + (n-1)\pi, \text{ for some } n \in \mathbb{Z}. \tag{2.9}$$

If an eigenvalue has geometric multiplicity 1 and a real eigenfunction, then all its real eigenfunctions share the same Prüfer angle. In this case, the Prüfer angle is called the *Prüfer angle* of the eigenvalue.

In the positive f case, the Prüfer angle characterization of eigenvalues for separated self-adjoint BCs is well known. In the indefinite f case, the indices of the eigenvalues for separated self-adjoint BCs are defined by using a similar characterization.

Lemma 1. *Assume that f changes sign on (a, b), and fix an $\alpha \in [0, \pi)$ and a $\beta \in (0, \pi]$. Then, for each $n \in \mathbb{Z}$, there is a unique eigenvalue for $\boldsymbol{S}_{\alpha,\beta}$, to be denoted by $\lambda_n = \lambda_n(\boldsymbol{S}_{\alpha,\beta})$, such that its Prüfer angle θ satisfies*

$$\theta(b, \lambda_n) = \beta + (n-1)\pi. \tag{2.10}$$

Moreover, $\lambda_n \to -\infty$ as $n \to -\infty$, and $\lambda_n \to +\infty$ as $n \to +\infty$.

Proof. See [1, Theorem 2.2]. □

Definition 1. For each $K \in \mathrm{SL}(2, \mathbb{R})$, define $\alpha_{0,K}, \alpha_{1,K} \in [0, \pi)$ by

$$\tan \alpha_{0,K} = -k_{12} / k_{11} \quad \text{or} \quad \cot \alpha_{0,K} = -k_{11} / k_{12}, \tag{2.11}$$

$$\tan \alpha_{1,K} = -k_{22} / k_{21} \quad \text{or} \quad \cot \alpha_{1,K} = -k_{21} / k_{22}. \tag{2.12}$$

For a fixed $\lambda_* \in \mathbb{R}$, we set

$$\Psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} := \Phi(b, \lambda_*), \tag{2.13}$$

and call it the *transfer matrix* of (1.1) with $\lambda = \lambda_*$; and $\alpha_{0,\Psi}$ and $\alpha_{1,\Psi}$ are further abbreviated as α_0, α_1, and β_0, respectively.

Note that α_0 is the zero of $\psi_{12} \cos\alpha + \psi_{11} \sin\alpha$ on the interval $[0, \pi)$, and α_1 is the zero of $\psi_{22} \cos\alpha + \psi_{21} \sin\alpha$.

Lemma 2. *Let* $\lambda_* \in \mathbb{R}$, *introduce* Ψ *and* α_0 *as in Definition* 1, *and define a function* $v : [0, \pi) \to (0, \pi]$ *by*

$$v(\alpha) = \operatorname{arc\,cot} \frac{\psi_{21} \sin \alpha + \psi_{22} \cos \alpha}{\psi_{11} \sin \alpha + \psi_{12} \cos \alpha}. \tag{2.14}$$

Then, the separated self-adjoint boundary conditions having λ_* *as an eigenvalue are*

$$\boldsymbol{S}_{\alpha, v(\alpha)},\ \alpha \in [0, \pi). \tag{2.15}$$

Moreover, there is an $n \in \mathbb{Z}$ *such that* n *and* $n+1$ *are the only two choices for the index of* λ_* *as an eigenvalue for* $\boldsymbol{S}_{\alpha, v(\alpha)}$: *The choice is* n *for* $0 \le \alpha \le \alpha_0$, *and* $n+1$ *for* $\alpha_0 < \alpha < \pi$.

It is understood that in (2.15), $u(\alpha_0) = \pi$.

Proof. The first claim is deduced from (4.8) in [8]. In the positive *f* case, the second claim comes from [3, Theorem 2.2], and the proof there also works for the indefinite *f* case. □

To give the indices of the eigenvalues for coupled self-adjoint BCs in the indefinite *f* case, we define

$$\mathcal{L}_1 = \{K \in \mathrm{SL}(2, \mathbb{R});\ k_{11} > 0,\ k_{12} \le 0\}, \tag{2.16}$$

$$\mathcal{L}_2 = \{K \in \mathrm{SL}(2, \mathbb{R});\ k_{11} \le 0,\ k_{12} < 0\}. \tag{2.17}$$

Note that for each $K \in \mathrm{SL}(2, \mathbb{R})$, either $K \in \mathcal{L}_1 \cup \mathcal{L}_2$ or $-K \in \mathcal{L}_1 \cup \mathcal{L}_2$.

Lemma 3. *Assume that f changes sign on* (a, b). *Fix* $\alpha\,\gamma \in (-\pi, \pi)$ *and a* $K \in \mathrm{SL}(2, \mathbb{R})$, *and set* $\boldsymbol{A}$ *to be the self-adjoint boundary condition given by* (2.7). *Let* $\{\mu_n;\ n \in \mathbb{Z}\}$ *be the eigenvalues for* $\boldsymbol{S}_{\alpha_{0,K}, \pi}$, *and denote by* $\{\nu_n;\ n \in \mathbb{Z}\}$ *the eigenvalues for* $\boldsymbol{S}_{\alpha_{1,K}, \pi/2}$.

(i) *If* $K \in \mathcal{L}L_1 \cup \mathcal{L}_2$, *then*: μ_n *is not an eigenvalue for* $\boldsymbol{A}$ *for any odd* $n \in \mathbb{Z}$; *and for each odd* $n \in \mathbb{Z}$, *there are exactly two eigenvalues* (*counting multiplicity*) *for* $\boldsymbol{A}$ *in the interval* (μ_n, μ_{n+2}), *to be denoted by* $\lambda_{n+1}(\boldsymbol{A})$ *and* $\lambda_{n+2}(\boldsymbol{A})$ *in non-decreasing order.*

(ii) *If* $\gamma = 0$ *and* $K \in \mathrm{SL}(2, \mathbb{R}) \setminus (\mathcal{L}_1 \cup \mathcal{L}_2)$, *then*: μ_n *is not an eigenvalue for* $\boldsymbol{A}$ *for any even* $n \in \mathbb{Z}$; *and for each even* $n \in \mathbb{Z}$, *there are exactly two eigenvalues* (*counting multiplicity*) *for* $\boldsymbol{A}$ *in the interval* (μ_n, μ_{n+2}), *to be denoted by* $\lambda_{n+1}(\mathbf{A})$ *and* $\lambda_{n+2}(\mathbf{A})$ *in non-decreasing order.*

In particular, there are infinitely many eigenvalues, unbounded from both below and above, for any coupled self-adjoint boundary condition.

Proof. The proof is similar with [2, Theorem 2.17]. □

The following theorem gives a simple solution of the index problem for eigenvalues for coupled self-adjoint BCs, and it is true for both the positive *f* case and the indefinite *f* case. Here, for any two objects c_1 and c_2, the notation $\{c_1, c_2\}$ with bold faced braces means each of c_1 and c_2.

Theorem 1. *Let* λ_* *be an eigenvalue for a coupled self-adjoint boundary condition* $\boldsymbol{A}$, *given by* (2.7) *with* $\gamma \in \mathbb{R}$ *and* $K \in \mathrm{SL}(2, \mathbb{R})$. *Introduce* Ψ, $\alpha_{0,K}$, α_0, $\alpha_{1,K}$, *and* α_1 *by Definition* 1.

(i) *We have that* $\alpha_{0,K} \neq \alpha_{1,K}$, *and* $\alpha_0 \neq \alpha_1$.

(ii) *The number* λ_* *is also an eigenvalue for the separated self-adjoint boundary condition* $\boldsymbol{S}_{\alpha_0, \pi}$. *Denote by* $n \in \mathbb{Z}$, *the corresponding index.*

(iii) *If* $\alpha_{0,K} < \alpha_0$, *then* λ_* *is a simple eigenvalue for* $\boldsymbol{A}$, *and its index is n.*

(iv) *If* $\alpha_{0,K} > \alpha_0$, *then* λ_* *is a simple eigenvalue for* $\boldsymbol{A}$, *and its index is* $n+1$.

(v) *If* $\alpha_{0,K} = \alpha_0$ *and* $\alpha_{1,K} = \alpha_1$, *then* λ_* *is a double eigenvalue for* $\boldsymbol{A}$, *and its indices are n and* $n+1$.

Proof. The proof is similar with [10, Theorem 1.16].

Note that $\alpha_{0,K} = \alpha_0$, iff $\alpha_{1,K} = \alpha_1$. □

Using Lemma 2, the solution of the index problem given in Theorem 1 can be restated as follows: If $\alpha_{0,K} < \alpha_0$, then λ_* is a simple eigenvalue for $\boldsymbol{A}$, and its index is equal to the index of λ_* as an eigenvalue for $\boldsymbol{S}_{\alpha, v(\alpha)}$ with any $\alpha \in [0, \alpha_0]$; if $\alpha_{0,K} > \alpha_0$, then λ_* is a simple eigenvalue for $\boldsymbol{A}$, and its index is equal to the index of λ_* as an eigenvalue for $\boldsymbol{S}_{\alpha, v(\alpha)}$ with any $\alpha \in (\alpha_0, \pi)$; if $\alpha_{0,K} = \alpha_0$ and $\alpha_{1,K} = \alpha_1$, then λ_* is a double eigenvalue for $\boldsymbol{A}$, and its smaller index is equal to the index of λ_* as an eigenvalue for $\boldsymbol{S}_{\alpha, v(\alpha)}$ with any $\alpha \in [0, \alpha_0]$. Therefore, the determination of the index or indices of an eigenvalue for a coupled self-adjoint BC is converted to finding the index of the same number as an eigenvalue for one separated self-adjoint BC.

A direct consequence of Theorem 1 is the following result.

Corollary 1. *Let* λ_*, $\boldsymbol{A}$, Ψ, α_0, *and* n *be the same as in Theorem* 1. *If* λ_* *is double for* $\boldsymbol{A}$, *then its indices are* n *and* $n + 1$.

The following result gives a complete description of the condition of $\alpha_{0,K} = \alpha_0$, and it also holds for both the positive *f* case and the indefinite *f* case.

Theorem 2. *Let* λ_*, γ, K, $\boldsymbol{A}$, Ψ, $\alpha_{0,K}$, *and* α_0 *be the same as above. Assume further that* γ *is in its standard range* $[0, \pi)$. *Then,* $\alpha_{0,K} = \alpha_0$, *if and only if* $\gamma = 0$ *and the first rows of* K *and* Ψ *are equal, i.e.*,

$$e^{i\gamma} K = K = \Psi. \tag{2.18}$$

Proof. The proof is similar with [10, Theorem 1.20]. □

3. Main Idea of the Numerical Algorithm

Given an eigenvalue λ_* for a coupled self-adjoint BC $\boldsymbol{A}$, Theorem 1 yields the following *algorithm for computing the index or indices of* λ_*.

Step 1. Normalize $\boldsymbol{A}$ into the form given in (2.7) with $\gamma \in \mathbb{R}$ and $K \in \mathrm{SL}(2, \mathbb{R})$; and compute $\alpha_{0,K}$, $\alpha_{1,K} \in [0, \pi)$ by (2.11) and (2.12).

Step 2. Approximate the fundamental solution matrix $\Phi(t, \lambda_*)$ for $a \le t \le b$ using

$$\Phi'(t, \lambda_*) = \begin{pmatrix} 0 & 1/f(t) \\ q(t) - \lambda_* w(t) & 0 \end{pmatrix} \Phi(t, \lambda_*), \quad \Phi(a, \lambda_*) = I; \tag{3.1}$$

define the transfer matrix Ψ via (2.13); and compute α_0, $\alpha_1 \in [0, \pi)$ by

$$\tan \alpha_0 = -\psi_{12} / \psi_{11} \quad \text{or} \quad \cot \alpha_0 = -\psi_{11} / \psi_{12}, \tag{3.2}$$

$$\tan \alpha_1 = -\psi_{22} / \psi_{21} \quad \text{or} \quad \cot \alpha_1 = -\psi_{21} / \psi_{22}. \tag{3.3}$$

Step 3. Let

$$\alpha_* = \begin{cases} (\alpha_0 + \pi)/2, & \text{if } \alpha_0 \le \pi/2, \\ \alpha_0/2, & \text{otherwise;} \end{cases} \tag{3.4}$$

and compute $\beta_* \in (0, \pi]$ by

$$\tan \beta_* = \frac{\psi_{11} \sin \alpha_* + \psi_{12} \cos \alpha_*}{\psi_{21} \sin \alpha_* + \psi_{22} \cos \alpha_*} \quad \text{or} \quad \cot \beta_* = \frac{\psi_{21} \sin \alpha_* + \psi_{22} \cos \alpha_*}{\psi_{11} \sin \alpha_* + \psi_{12} \cos \alpha_*}. \tag{3.5}$$

Step 4. Use $\Phi(t, \lambda_*)$ to form an eigenfunction y_* for λ_* as an eigenvalue for the separated BC $\boldsymbol{S}_{\alpha_*, \beta_*}$; and compute the difference $\theta_*(b, \lambda_*) - \beta_*$. Denote the difference by $n\pi$, if $\alpha_0 \le \pi/2$, and by $(n-1)\pi$ otherwise.

Step 5. Use Theorem 1 to determine the index or indices of λ_* as an eigenvalue for $\boldsymbol{A}$: The index is n, if $\alpha_{0,K} < \alpha_0$; the index is $n+1$, if $\alpha_{0,K} > \alpha_0$; and the indices are n and $n+1$, if $\alpha_{0,K} = \alpha_0$ and $\alpha_{1,K} = \alpha_1$.

4. Numerical Examples

Example 1. Let $q(t) = t - 1,$ and $w(t) = t.$ Then, 1 is an eigenvalue of the SLP

$$-y'' + qy = \lambda wy \text{ on } (0, 7\pi/4), \quad Y(7\pi/4) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix} Y(0), \tag{4.1}$$

with an eigenfunction $y(t) = \cos t.$ It is direct to verify that $\sin t$ is another solution of the SLE in (4.1) with $\lambda = 1,$ and hence

$$\Phi(t, 1) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad \Psi = \Phi(7\pi/4, 1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{4.2}$$

Note that $\sin t$ has exactly 1 zero in $(0, 7\pi/4).$ Thus, the index of 1 as an eigenvalue for $\boldsymbol{S}_{0,\beta_0}$ is 2. By [Example 3.7, 9],

$$0.24995, \quad 0.90578, \quad 0.99999, \quad 2.04145, \quad 2.61244, \tag{4.3}$$

are approximations of the eigenvalues in the interval $[-4, 4]$.

We have

$$\alpha_0 = \pi/4 \approx 0.7854 < \alpha_{0,K} \approx 1.1072. \tag{4.4}$$

Thus, $\lambda_3 = 1$ by Theorem 1. Actually, the codes yield their indices:

$$\lambda_1 \approx 0.24995, \quad \lambda_2 \approx 0.90578, \quad \lambda_3 \approx 0.99999, \tag{4.5}$$

$$\lambda_4 \approx 2.04145, \quad \lambda_5 \approx 2.61244.$$

If the BC chosen is

$$Y(7\pi/4) = \begin{pmatrix} 1/\sqrt{2} & -\sqrt{2}/(1+\sqrt{3}) \\ 1/\sqrt{2} & \sqrt{6}/(1+\sqrt{3}) \end{pmatrix} Y(0). \tag{4.6}$$

By [Example 3.7, 9],

$$0.28841, \quad 1.0, \quad 1.03071, \quad 2.30948, \quad 2.61988, \quad 4.63613, \tag{4.7}$$

are approximations of the eigenvalues in the interval $[-5, 5]$.

We have

$$\alpha_0 = \pi / 4 \approx 0.7854 > \alpha_{0,K} \approx 0.7321. \tag{4.8}$$

Thus, $\lambda_2 = 1$ by Theorem 1. Actually, the codes yield their indices:

$$\lambda_1 \approx 0.28841, \quad \lambda_2 \approx 1.0, \quad \lambda_3 \approx 1.03071, \tag{4.9}$$

$$\lambda_4 \approx 2.30948, \quad \lambda_5 \approx 2.61988, \quad \lambda_6 \approx 4.63613. \tag{4.10}$$

If we change the BC to

$$Y(7\pi / 4) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} Y(0). \tag{4.11}$$

By [Example 3.7, 9],

$$0.27973, \quad 0.99985, \quad 2.23403, \quad 2.61682, \quad 4.52107, \tag{4.12}$$

are approximations of the eigenvalues in the interval $[-5, 5]$.

We have

$$\alpha_0 = \alpha_{0,K} = \pi / 4, \quad \alpha_1 = \alpha_{1,K} = 3\pi / 4. \tag{4.13}$$

Thus, $\lambda_2 = \lambda_3 = 1$ by Theorem 1. Actually, the codes yield their indices:

$$\lambda_1 \approx 0.27973, \quad \lambda_2 = \lambda_3 \approx 0.99985, \quad \lambda_4 \approx 2.23403, \tag{4.14}$$

$$\lambda_5 \approx 2.61682, \quad \lambda_6 \approx 4.52107. \tag{4.15}$$

The rest of examples, we always set $f(t) = \operatorname{sgn} t$.

Example 2. Let $q(t) = t^2 - 9 \operatorname{sgn} t$, and $w(t) = t^2$. Then, 1 is an eigenvalue of the SLP

$$-(fy')' + qy = \lambda wy \text{ on } (-3\pi / 2, \pi / 4), \quad Y(\pi / 4) = KY(-3\pi / 2), \tag{4.16}$$

with an eigenfunction $y(t) = \operatorname{sgn} t \sin(3t)$, where

$$K = \sqrt{2}\begin{pmatrix} 1/2 & -1/[3(\sqrt{3}-1)] \\ -3/2 & \sqrt{3}/(\sqrt{3}-1) \end{pmatrix}. \tag{4.17}$$

It is direct to verify that $\cos(3t)$ is an another solution of the SLE in (4.16) with $\lambda = 1,$ and hence

$$\Phi(t, 1) = \begin{pmatrix} \operatorname{sgn} t \sin(3t) & -\cos(3t)/3 \\ 3\cos(3t) & \operatorname{sgn} t \sin(3t) \end{pmatrix}, \tag{4.18}$$

$$\Psi = \phi(\pi/4, 1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1/3 \\ -3 & 1 \end{pmatrix}, \tag{4.19}$$

with an eigenfunction $y(t) = \cos t.$ By [Example 4.2, 9], the index of 1 as an eigenvalue for $\boldsymbol{S}_{0,\beta_0}$ is –3, and

$$0.55262, \quad 1.0, \quad 1.52118, \quad 1.71224, \quad 2.80841, \quad 9.72016, \tag{4.20}$$

are approximations of the eigenvalues in the interval [0, 12].

We have

$$\alpha_0 \approx 2.8198 < \alpha_{0,K} \approx 0.4448. \tag{4.21}$$

Thus, $\lambda_{-3} = 1$ by Theorem 1. Actually, the codes yield their indices:

$$\lambda_{-4} \approx 0.55262, \quad \lambda_{-3} \approx 1.0, \quad \lambda_{-2} \approx 1.52118, \tag{4.22}$$

$$\lambda_{-1} \approx 1.71224, \quad \lambda_0 \approx 2.80841, \quad \lambda_1 \approx 9.72016.$$

Example 3. Let $q(t) = t^2 - \operatorname{sgn} t,$ and $w(t) = t^2.$ Then, 1 is an eigenvalue of the SLP

$$-(fy')' + qy = \lambda wy \text{ on } (-3\pi/2, \pi/4), \quad Y(\pi/4) = KY(-3\pi/2), \tag{4.23}$$

with an eigenfunction $y(t) = \operatorname{sgn} t \sin t,$ where

$$K = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}. \tag{4.24}$$

It is direct to verify that cos (t) is an another solution of the SLE in (4.16) with $\lambda = 1,$ and hence

$$\Phi(t, 1) = \begin{pmatrix} -\operatorname{sgn} t \sin t & \cos t \\ -\cos t & -\operatorname{sgn} t \sin t \end{pmatrix}, \tag{4.25}$$

$$\Psi = \Phi(\pi / 4, 1) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}. \tag{4.26}$$

By [Example 4.8, 9], the index of 1 as an eigenvalue for $\boldsymbol{S}_{0,\beta_0}$ is –1, and

$$0.14785, \quad 0.37465, \quad 1.0, \quad 2.23275, \tag{4.27}$$

are approximations of the eigenvalues in the interval [0, 6].

We have

$$\alpha_0 = \alpha_{0,K} = \pi / 4, \quad \alpha_1 = \alpha_{1,K} = 3\pi / 4. \tag{4.28}$$

Thus, $\lambda_{-1} = \lambda_0 = 1$ by Theorem 1. Actually, the codes yield their indices:

$$\lambda_{-3} \approx 0.14785, \quad \lambda_{-2} \approx 0.37465, \quad \lambda_{-1} = \lambda_0 \approx 1.0, \quad \lambda_1 \approx 2.23275.$$

Example 4. Let $q(t) = t^2 - 25 \operatorname{sgn} t,$ and $w(t) = t^2$. Then, 1 is an eigenvalue of the SLP

$$-(fy')' + qy = \lambda wy \text{ on } (-3\pi / 2, \pi / 4), \quad Y(\pi / 4) = KY(-3\pi / 2), \tag{4.29}$$

with an eigenfunction $y(t) = \operatorname{sgn} t \sin(5t),$ where

$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -2 / 5 \\ 5 & 0 \end{pmatrix}. \tag{4.30}$$

It is direct to verify that $\cos(5t)$ is an another solution of the SLE in (4.29) with $\lambda = 1,$ and hence

$$\Phi(t, 1) = \begin{pmatrix} -\operatorname{sgn} t \sin(5t) & \cos(5t) / 5 \\ -5 \cos(5t) & -\operatorname{sgn} t \sin(5t) \end{pmatrix}, \tag{4.31}$$

$$\Psi = \Phi(\pi / 4, 1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1/5 \\ 5 & 1 \end{pmatrix}. \tag{4.32}$$

By [Example 4.15, 9], the index of 1 as an eigenvalue for $\mathcal{S}_{0,\beta_0}$ is –6, and

$$-1.69784, \quad -1.17353, \quad 0.71166, \quad 0.99999, \quad 2.21812, \quad 2.44785, \tag{4.33}$$

are approximations of the eigenvalues in the interval $[-3, 3]$.

We have

$$\alpha_0 \approx 0.1974 < \alpha_{0,K} \approx 0.3805. \tag{4.34}$$

Thus, $\lambda_{-5} = 1$ by Theorem 1. Actually, the codes yield their indices:

$$\lambda_{-8} \approx -1.69784, \quad \lambda_{-7} \approx -1.17353, \quad \lambda_{-6} \approx 0.71166, \tag{4.35}$$

$$\lambda_{-5} \approx 0.99999, \quad \lambda_{-4} \approx 2.21812, \quad \lambda_{-3} \approx 2.44785.$$

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